A model theoretic approach to sparsity

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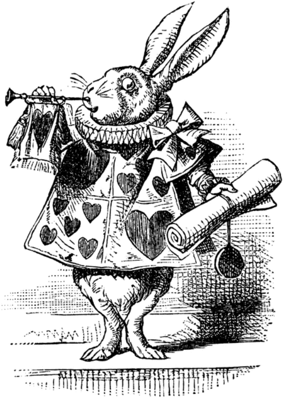
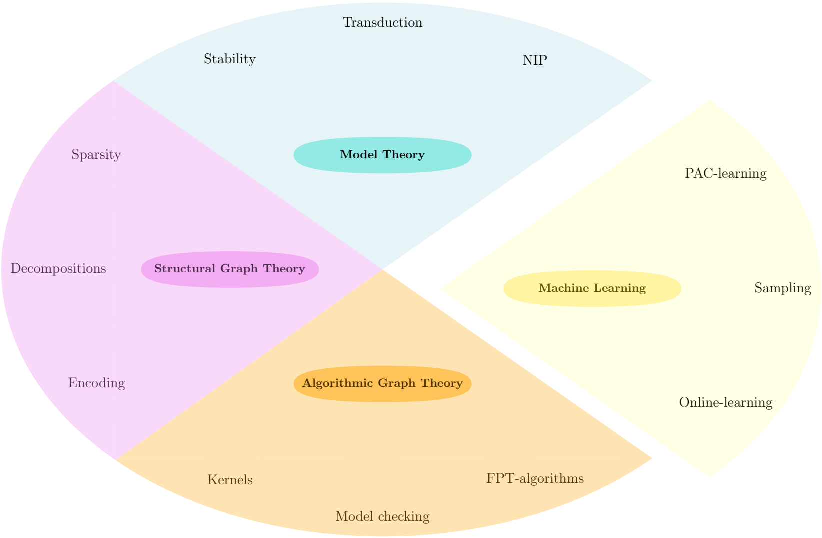
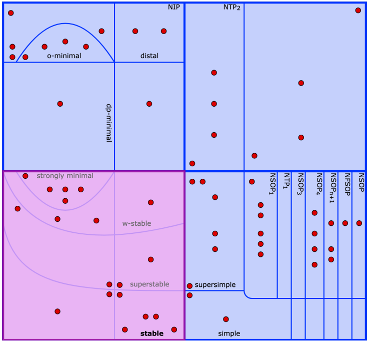
Introduction



General View



Model theory vs Graph theory



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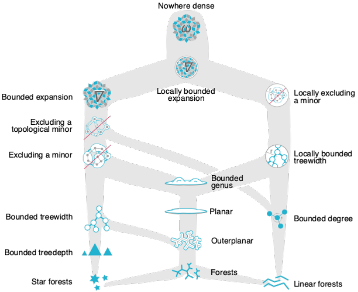

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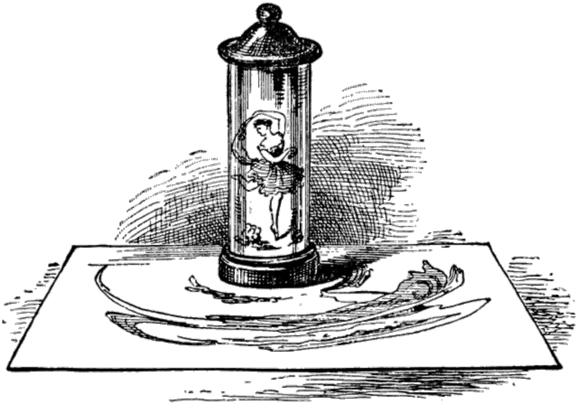
Transductions



Transductions

How to encode graphs in a structure?

- Use a formula $\varphi(x, y)$ to define the edges,
- Use colors to encode several graphs in a same graph,
- Extract induced subgraphs.

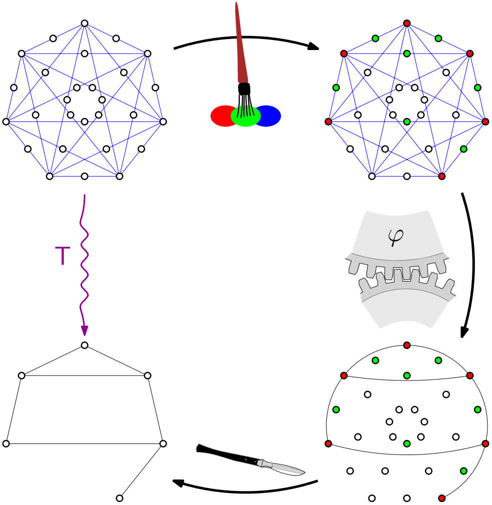
$$\mathcal{C} \twoheadrightarrow \mathcal{D}$$

Remark

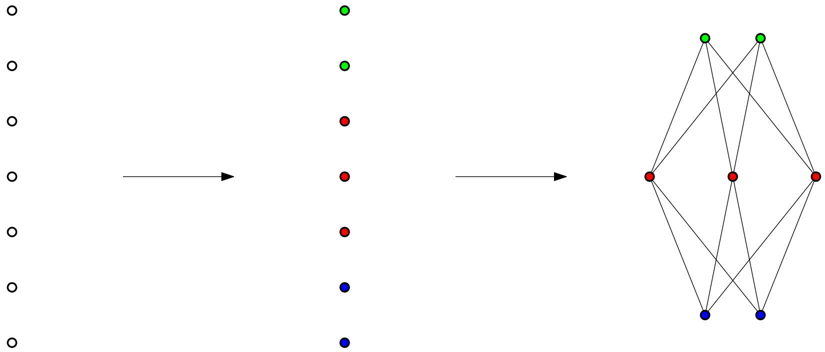
Transduction compose. In particular,

$$\mathcal{C} \twoheadrightarrow \mathcal{D} \twoheadrightarrow \mathcal{E} \quad \Longrightarrow \quad \mathcal{C} \twoheadrightarrow \mathcal{E}$$

Transduction: Color, Compute, and Cut

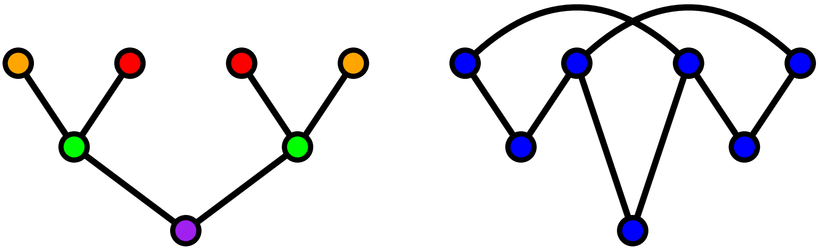


Example 1: from edgeless graphs



Edgeless ⟶ Blowing of a fixed graph

Example 2: from bounded height trees

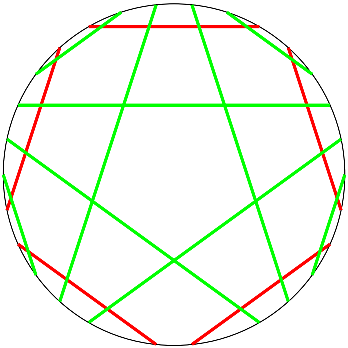


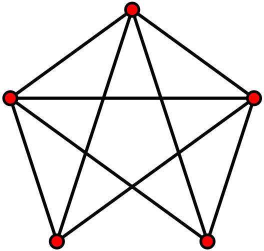
Bounded height trees $\twoheadrightarrow$ Bounded shrub-depth

More:

$\mathcal{C}$ has bounded shrub-depth $\iff$ $(\exists n)\ \mathcal{Y}_n \twoheadrightarrow \mathcal{C}$

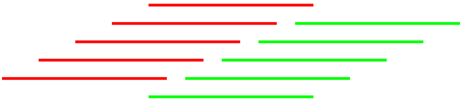
Example 3: from circle graphs

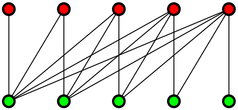




Interval graphs $\twoheadrightarrow$ All graphs

Example 4: from unit interval graphs



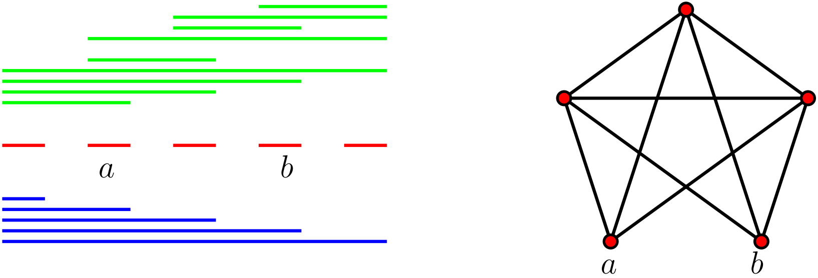


Unit interval graphs $\longrightarrow\!\!\!\!\twoheadrightarrow$ Half-graphs

Problem

Can we get all graphs? Probably not!

Example 5: from interval graphs



Interval graphs $\twoheadrightarrow$ All graphs

Monadic dependence and stability

Monadically NIP	Monadically Stable
Every definable class in a monadic lift has bounded VC-dimension	Every definable class in a monadic lift has bounded Littlestone dimension
No monadic lift can interpret all element-set graphs	No monadic lift can interpret all half graphs
$\mathcal{C} \not\twoheadrightarrow \mathcal{G}$	$\mathcal{C} \not\twoheadrightarrow \mathsf{LO}$

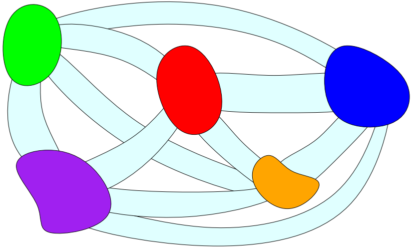
(Baldwin, Shelah '85)

Monadic dependence and stability

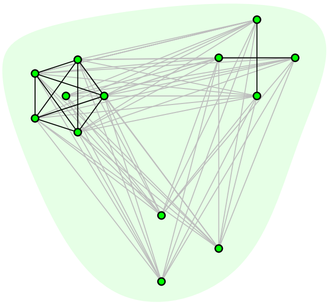
- Circle graphs are not monadically NIP.
- Interval graphs are not monadically NIP.
- Unit interval graphs are not monadically stable (but monadically NIP?).
- Cographs are not monadically stable, but monadically NIP.
- Every transduction of a bounded expansion class is monadically stable (Adler & Adler '14).

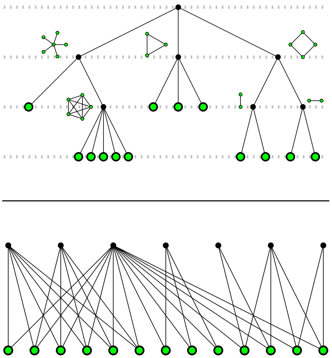
Sparsification & Decomposition

Sparsification

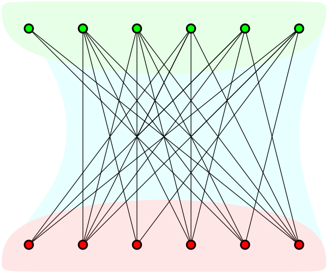


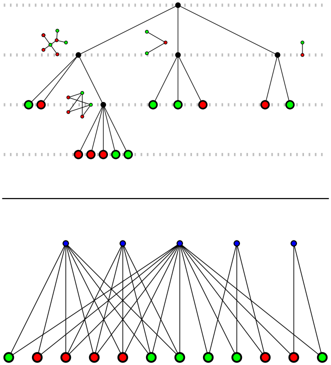
Vertex bloc: bounded depth cographs



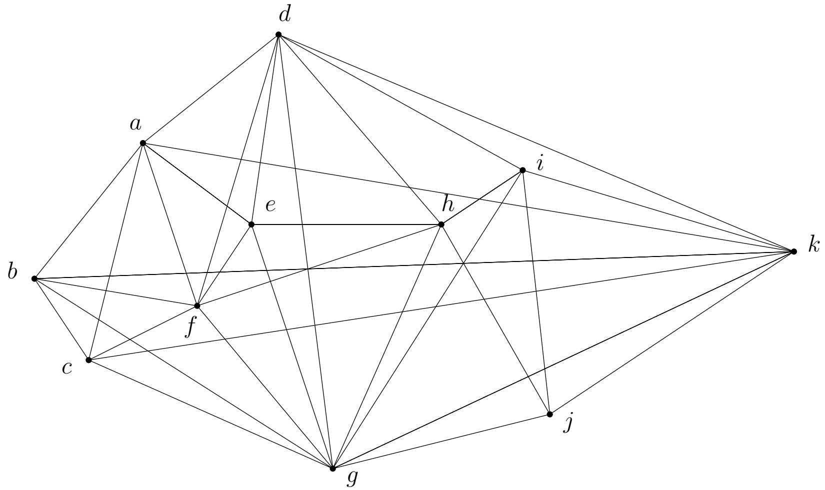


Edge bloc: bounded depth bi-cographs

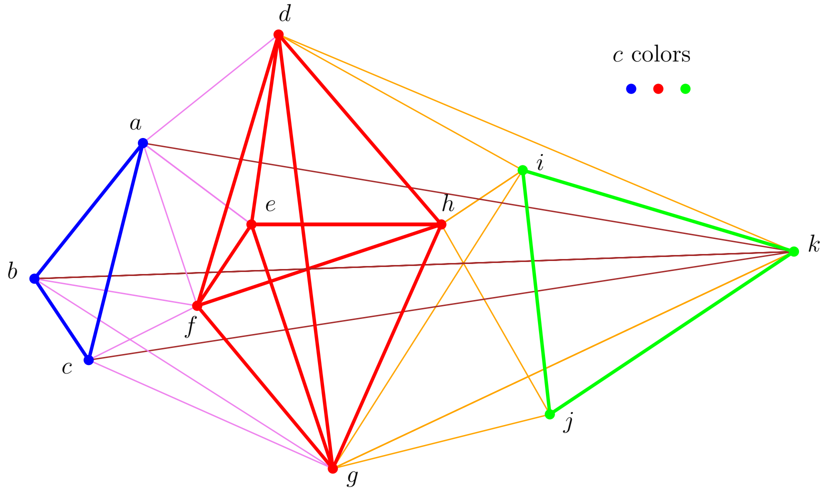




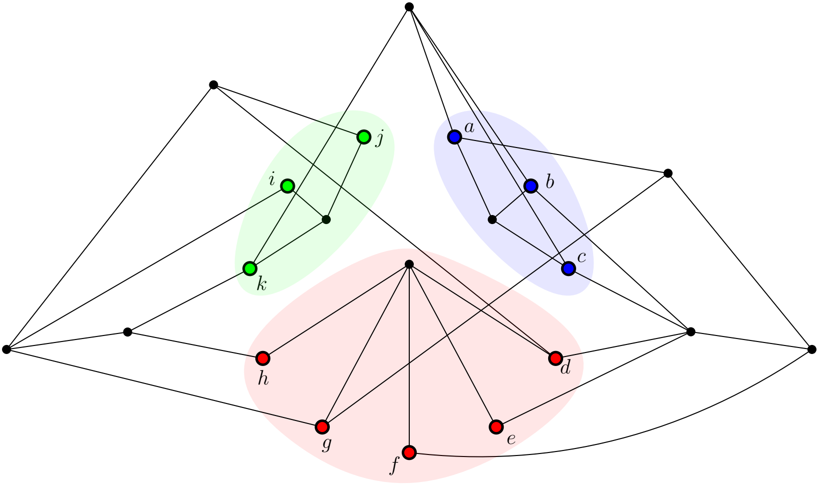
(c, d)-fold coloring



(c, d)-fold coloring



Sparsification: Cut & Paste



Structural Sparsity

Theorem (Gajarský, Kreutzer, Kwon, Nešetril, POM, Pilipczuk, Siebertz, Toruńczyk '18)

For a class of graphs $\mathcal{C}$ with (c, d)-fold coloring the following are equivalent:

- $\mathcal{C}$ has low shrub-depth decompositions
- Sparsify($\mathcal{C}$) has tree-depth decompositions;
- Sparsify($\mathcal{C}$) has bounded expansion.
- $\mathcal{C}$ has structurally bounded expansion;

If (c, d)-fold colorings can be computed in time $F(n)$ for $G \in \mathcal{C}$ then checking a first-order sentence ϕ on $\mathcal{C}$ can be done in time

$$F(n) + C(\phi, \mathcal{C})n.$$

Decompositions

Low rank-width decomposition $\Longrightarrow$ χ-bounded

(Kwon, Pilipczuk, Siebertz '17) (Dvořák, Kral' '12)

$\Uparrow$

Low linear rank-width decomposition

Monadically NIP?

$\Uparrow$

SBE $\Longleftrightarrow$ Low shrub-depth decomposition $\Longrightarrow$ linearly χ-bounded

$\Uparrow$

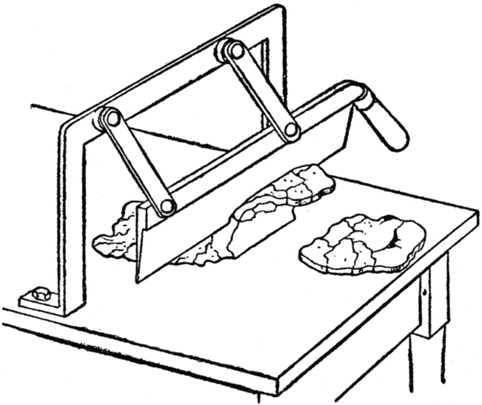
BE $\Longleftrightarrow$ Low tree-depth decomposition

Monadically stable

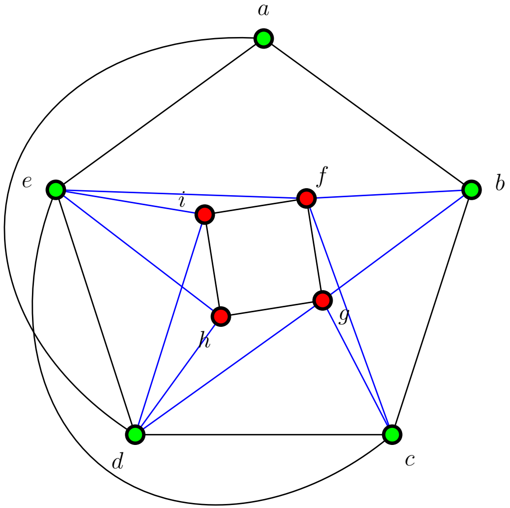
Decompositions (examples)

- Interval graphs do not have low rank-width decomposition (Kwon, Pilipczuk, Siebertz '17)
- Unit interval graphs have low rank-width decomposition but unbounded rank-width (Golumbic, Rotics '99 and Kwon, Pilipczuk, Siebertz '17)
- Cographs have bounded rank-width but no low linear rank-width decomposition
- Half-graphs have bounded linear rank-width but have no low shrub-depth decomposition (follows from Adler[2] '14 and Gajarský et al. '18).

Rank-width



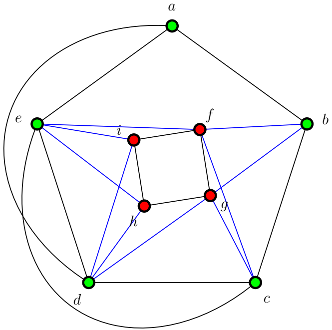
Cut-rank

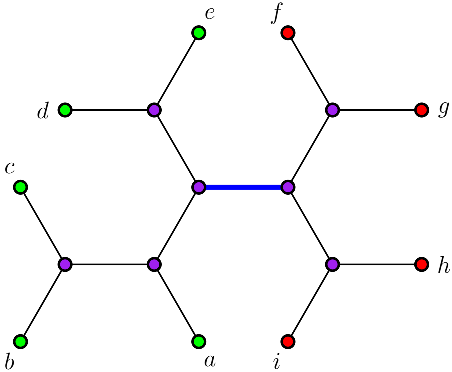


	a	b	c	d	e
f	0	1	1	0	1
g	0	1	1	1	0
h	0	0	0	1	1
i	0	0	0	1	1

	a	b	c	d	e
f	0	1	1	0	1
g	0	1	1	1	0
h	0	0	0	1	1
i	0	0	0	1	1

Rank-width





Rank-width and Linear rank-width

Rank-width	Linear rank-width
Subcubic rank-decomposition tree with bounded width	Caterpillar rank-decomposition tree with bounded width
$\mathsf{TO} \twoheadrightarrow \mathcal{C}$	$\mathsf{LO} \twoheadrightarrow \mathcal{C}$

(Colcombet '07)

Remark

Bounded rank-width implies monadically NIP.

Rank-width and Linear rank-width

Intuitively:

Dense		Sparse
Rank-width	$\longleftrightarrow$	Treewidth
Linear rank-width	$\longleftrightarrow$	Pathwidth
?	$\longleftrightarrow$	Bandwidth
Shrub-depth	$\longleftrightarrow$	Tree-depth

Rank-width and Linear rank-width

On reflection. . .

Monadically NIP $\mathcal{C} \not\twoheadrightarrow \mathcal{G}$		Monadically Stable $\mathcal{C} \not\twoheadrightarrow \mathsf{LO}$		Sparse
Rank-width	$\leftrightarrow$	T(Treewidth)	$\leftrightarrow$	Treewidth
Linear rank-width	$\leftrightarrow$	T(Pathwidth)	$\leftrightarrow$	Pathwidth
		T(Path)	$\leftrightarrow$	Bandwidth
		Shrub-depth	$\leftrightarrow$	Tree-depth

Restricted Dualities

Assume $\mathsf{LO} \twoheadrightarrow \mathcal{C}$. Is it true that

$$\mathcal{C} \not\twoheadrightarrow \mathsf{LO} \iff (\exists n)\ \mathcal{PW}_n \twoheadrightarrow \mathcal{C}\ ?$$

Assume $\mathsf{TO} \twoheadrightarrow \mathcal{C}$. Is it true that

$$\mathcal{C} \not\twoheadrightarrow \mathsf{LO} \iff (\exists n)\ \mathcal{TW}_n \twoheadrightarrow \mathcal{C}\ ?$$

Rank-width and stability

Theorem (Nešetřil, POM, Rabinovich, Siebertz '19+)

Let $\mathcal{C}$ be a class of graphs. The following are equivalent:

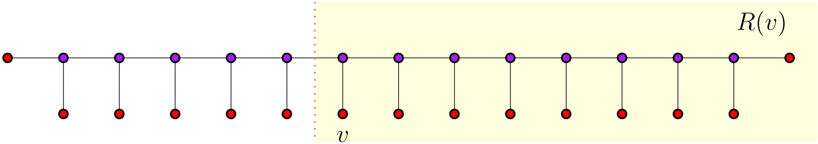
1. $\mathcal{C}$ has bounded linear rank-width and excludes some semi-induced half-graph,
2. $\mathcal{C}$ is a transduction of a class with bounded pathwidth.

Corollary

Let $\mathcal{C}$ be a class of graphs. The following are equivalent:

1. $\mathcal{C}$ is monadically stable and has low linear rank-width decompositions,
2. $\mathcal{C}$ has structurally bounded expansion.

Linear rank width



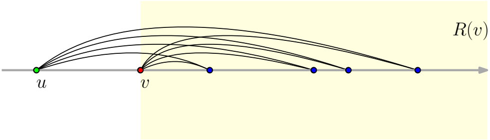
There exists a linear order $<$ with the property that for every $v \in V$

$$\begin{aligned} & \operatorname{cutrk}(V - R(v), R(v)) \leq r \\ \iff \quad & \dim_{\mathbb{Z}_2}\left(\left\{N(u) \cap R(v) \mid u < v\right\}\right) \leq r \\ \implies \quad & \left|\left\{N(u) \cap R(v) \mid u < v\right\}\right| \leq 2^r \end{aligned}$$

Types of vertices

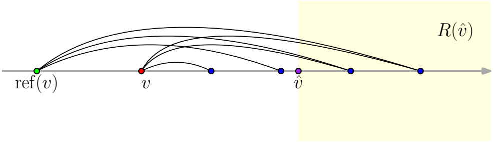
$$(\exists u < v)\ N(u) \cap R(v) = N(v) \cap R(v)$$

Inactive vertex v

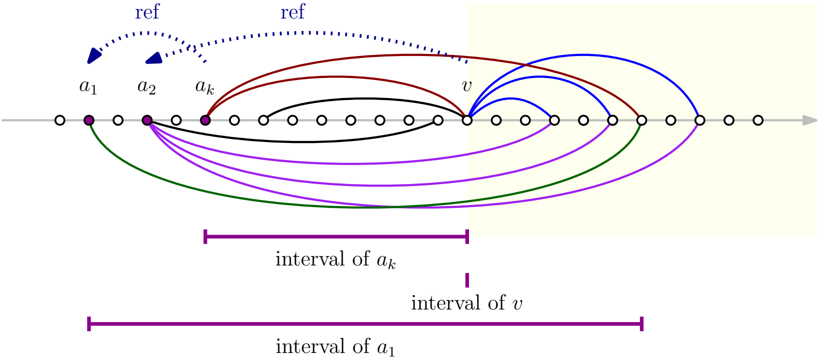


$$N(\mathrm{ref}(v)) \cap R(\hat{v}) = N(v) \cap R(\hat{v})$$

Active vertex v



Activity intervals



Coding

- Color the intersection graph of activity intervals with $2^r + 1$ colors $\rightsquigarrow \gamma(v)$.
- Let $\text{class}(v) = (\gamma(\text{ref}(v)), \gamma(v))$.
- Link v to all the $\leq 2^r$ vertices active at v and encode adjacency to them, and which of them is $\text{ref}(v)$.

Then if $x < y$ we have $xy \in E(G)$ if and only if

- either y is in the activity interval of x and the code of y indicates that y is adjacent to x,
- or y is not in the activity interval of x and $\text{ref}(x)$ is adjacent to y.

And so...

Problem

Assume	y is not in the activity interval of x
and	x is not in the activity interval of y.

How to determine wether $x < y$ or $y < x$?

And so...

Problem

Assume y is not in the activity interval of x
and x is not in the activity interval of y.

How to determine wether $x < y$ or $y < x$?

1. Only matters if adjacency of $\mathrm{ref}(x)$ and y is different from adjacency of $\mathrm{ref}(y)$ and x.

And so. . .

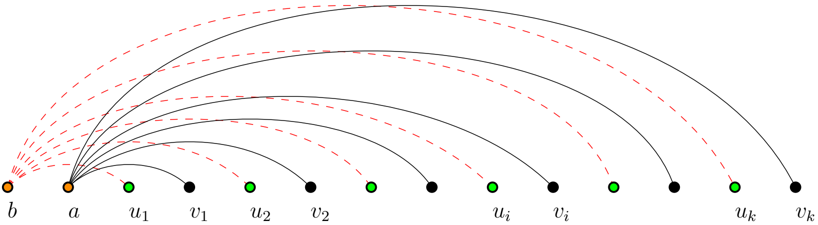
Problem

Assume y is not in the activity interval of x
and x is not in the activity interval of y.

How to determine wether $x < y$ or $y < x$?

1. Only matters if adjacency of $\mathrm{ref}(x)$ and y is different from adjacency of $\mathrm{ref}(y)$ and x.
2. For every a cut intervals $\{v \mid \mathrm{ref}(v) = a\}$ into sub-intervals corresponding to alternations and hard-code order between sub-intervals.

Difficult case



$$\mathrm{ref}(u_i) = a \quad \text{and} \quad \mathrm{ref}(v_i) = b.$$

$$\rightsquigarrow \quad u_i v_j \in E(G) \iff i \leq j.$$

More?

Conjecture

Let $\mathcal{C}$ be a class of graphs. The following are equivalent:

1. $\mathcal{C}$ has bounded linear rank-width and is monadically stable,
2. $\mathcal{C}$ is a transduction of a class with bounded treewidth.

If true, the following are equivalent:

1. $\mathcal{C}$ is monadically stable and has low rank-width decompositions,
2. $\mathcal{C}$ has structurally bounded expansion.

Full Dualities?

Conjecture

A class of graphs $\mathcal{C}$ has bounded shrub-depth if and only there is no surjective transduction from $\mathcal{C}$ to the class of all finite paths.

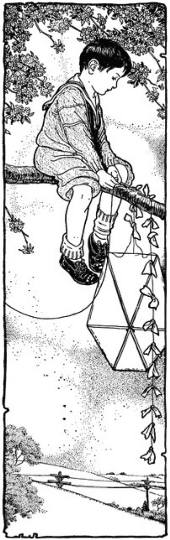
This would corresponds to a duality between bounded height trees and paths:

$$\mathcal{C} \not\twoheadrightarrow \mathcal{P} \quad \iff \quad (\exists n)\ \mathcal{Y}_n \twoheadrightarrow \mathcal{C}$$

Remark

It is well known that $\mathcal{Y}_n \not\twoheadrightarrow \mathcal{P}$. Hence

$$\mathcal{C} \not\twoheadrightarrow \mathcal{P} \quad \Longleftarrow \quad (\exists n)\ \mathcal{Y}_n \twoheadrightarrow \mathcal{C}$$



Thank you for your attention.